On Equal Cost Sharing in the Provision of an Excludable Public Good[1]

JORDI MASSÓ ANTONIO NICOLÒ ARUNAVA SEN

TRIDIB SHARMA LEVENT ÜLKÜ[2]

June 2013

Abstract: We study the efficiency and fairness properties of the equal cost sharing mechanism in the provision of a binary and excludable public good. According to the maximal welfare loss criterion, equal cost sharing is optimal within the class of strategyproof, individually rational and no-budget-deficit mechanisms only when there are 2 agents. In general the equal cost

[1]Acknowledgements: We thank María Angeles de Frutos, Rajat Deb, Emilio Gutiérrez, Rich McLean, Hervé Moulin, and William Thomson for their comments. We are especially grateful to Shinji Ohseto for his comments and helpful suggestions. We are grateful to an Associate Editor and three referees of this journal for many invaluable comments and suggestions. Jordi Massó acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centers of Excellence in R&D (SEV-2011-0075) and FEDER grant ECO2008-04756 (Grupo Consilidado-C), and from the Generalitat de Catalunya, through the prize "ICREA Academia" for excellence in research and grant SGR2009-419. He also acknowledges the support of MOVE, where he is an affiliated researcher. Antonio Nicolò's work is partially supported by the project "Intelligent Preference reasoning for multi-agent decision making" (Univ. of Padua).

[2]Jordi Massó is at the Universitat Autònoma de Barcelona and Barcelona GSE. Departament d'Economia i Història Economica, Edifici B, UAB. 08193 Bellaterra (Barcelona), Spain. Email: jordi.masso@uab.es.

Antonio Nicolò is at the Dipartimento di Scienze Economiche. Università degli Studi di Padova. Via del Santo 33. 35123, Padova, Italy. Email: antonio.nicolo@unipd.it.

Arunava Sen is at the Indian Statistical Institute. 7, S.J.S. Sansanwal Marg, New Delhi 110016, India. Email: asen@isid.ac.in

Tridib Sharma is at the Centro de Investigación Económica, ITAM, Av. Santa Teresa No. 930 Col. Héroes de Padierna, C.P. 10700 Mexico D.F. Email: sharma@itam.mx

Levent Ülkü is at the Centro de Investigación Económica, ITAM, Av. Santa Teresa No. 930 Col. Héroes de Padierna, C.P. 10700 Mexico D.F. Email: levent.ulku@itam.mx.

sharing mechanism is no longer optimal in this class: we provide a class of mechanisms obtained by symmetric perturbations of equal cost sharing with strictly lower maximal welfare loss. We show that if one of two possible fairness conditions is additionally imposed, equal cost sharing mechanism regains optimality.
JEL Classification: D71
Keywords: Excludable public good, Equal cost sharing, Maximal welfare loss

1 Introduction

An excludable and non-rivalrous binary public good is a good that can be used by several agents in a group, with the possibility of excluding some agents from its consumption. However once an agent gains access to the public good his valuation is independent of who else has access to it.[3] A group of agents, each privately informed about his valuation of the good, has to jointly decide on its provisioning. The good costs a fixed amount to produce independent of the number of users. Two decisions have to be made: (i) the set of users if the public good is provided, and (ii) the list of agents' contributions (or individual prices).

An example of the kind of problem we have in mind is the provision of online classes. Imagine a world-renowned scientist who is offered to teach several advanced online lectures in his area of expertise. The lectures would be recorded and students would be able to follow them at any time and any place. Of course, one would first need to enroll by paying a price to be able to virtually attend these lectures, which takes the form of, say, visiting a website and entering an individualized password. In this sense exclusion is feasible and, perhaps more importantly, implementable. As importantly, an enrolled student's value does not depend on who else has access to the lectures, hence there are no allocation externalities. In other words, following the lectures does not entail rivalry in consumption. Of course teaching takes time and its cost on the lecturer would typically depend on the size of his class in a lecture hall. But in the online context the time value of the lecturer would be independent of who he is virtually addressing, as all he would have to do would be to lecture in front of a camera. Hence his compensation could reasonably be designed independent of the users of the service he provides. Now (i) who among the grand set of potential students should have access to these online lectures, and (ii) how should they share the compensation of the lecturer?

These decisions will typically depend on agents' valuations; however, since these are private information they have to be elicited from the agents. In other words, the mechanism that maps profiles of valuations into allocations or decisions must be *incentive compatible.* We impose the requirement of *strategyproofness* that guarantees that no agent can gain by misrepresenting

[3] For instance, Deb and Razzolini (1999a, 1999b), Dobzinski et al. (2008), Moulin and Shenker (2001), Mutuswami (2005, 2008), Osheto (2000, 2005), Olszewski (2004), and Yu (2007).

his true valuation irrespective of his beliefs about the valuations of other agents. A second basic requirement is that the mechanism be *individually rational*, i.e. an agent who is a user cannot be charged more than his valuation while a non-user cannot make a positive payment. This requirement ensures that all agents participate voluntarily in the decision-making process. Finally, we shall require that there be *no-budget-deficit*, i.e. agents' contributions should cover the cost of provisioning the public good.

A simple and attractive mechanism that can be used to provide an excludable binary public good is the *equal cost sharing* mechanism. It can be implemented by auction-like indirect mechanisms.[4] At every profile of valuations, the equal cost sharing mechanism selects the allocation that maximizes (in the set-inclusion sense) the group of users subject to the following requirements: (i) each user's contribution is the cost of provision divided by the number of users, (ii) this contribution is no greater than his valuation, and (iii) all non-users pay zero. It is easy to verify that this mechanism is well-defined at every profile and satisfies strategy-proofness, individual rationality and no-budget-deficit. Our goal, in this paper, is to throw additional light on this mechanism. In particular, we investigate notions of *efficiency* and *fairness* associated with this mechanism.

It is well-known that strategyproofness, individual rationality and no-budget-deficit are incompatible with efficiency. Since there is no rivalry in consumption, excluding an agent with a strictly positive valuation is not efficient when aggregate valuation is above cost. Since individual rationality requires that the contribution paid by each agent is not larger than his valuation of the public good, agents will have incentives to mis-report their own valuation in order to lower their contribution. As a way out one could reduce prices, possibly all the way to zero. But then one would violate the requirement of no-deficit. In view of this incompatibility, a second-best approach is increasingly being adopted in the literature on mechanism design. This approach identifies a mechanism that minimizes the *maximal welfare loss* in the class of all strategyproof and individually rational mechanisms.[5]

---

[4] See Deb and Razzolini (1999a, 1999b).

[5] The worst-case welfare objective function is a well-established and widely-used criterion. For applications in related areas, see Moulin and Shenker (2001), Moulin (2008), and Juarez (2008a, 2008b) in the context of public good provision. See also Koutsoupias and Papadimitriou (1999), Roughgarden (2002), and Roughgarden and Tardos (2002) in the computer science literature on the price of anarchy, introduced to measure the effects of selfish routing in a congested network.

The welfare loss of a mechanism at a profile of valuations is the difference between the aggregate welfare of the first-best and the aggregate welfare of the mechanism evaluated at the profile. The maximal welfare loss of a mechanism is the supremum, taken over all profiles of valuations, of its welfare loss. Then, each mechanism is evaluated according to its maximal welfare loss and the goal is to select a mechanism that minimizes it. Since we are interested in mechanisms satisfying individual rationality, it turns out that inefficiencies arise from the exclusion (as users) of some (or all) agents who have strictly positive valuations. The maximal welfare loss of a mechanism is then the sum of the valuations of all non-users of the good at the preference profile which maximizes this sum.

We show that when there are two agents, the equal cost sharing mechanism minimizes maximal welfare loss in the class of all strategyproof, individually rational and no-budget-deficit mechanisms. However this result does not hold in general: we construct a class of feasible mechanisms which outperform equal cost sharing in terms of maximal welfare loss. However, the equal cost sharing mechanism satisfies certain additional fairness properties, which fail in the class we construct. We identify two such properties which we call *weak demand monotonicity* and *weak free entry.* When there are more than two agents, these two properties come into conflict with our second-best notion of efficiency. In the class of all strategy-proof, individually rational and no-deficit mechanisms which satisfy weak demand monotonicity *or* weak free entry, we show that the equal cost sharing mechanism minimizes maximal welfare loss. Thus, if one wants to improve upon the level of "efficiency" generated by the equal cost sharing mechanism, one has to give up on certain notions of fairness.

We say that a mechanism violates weak free entry if there is a type profile where an agent with a lower evaluation of the good is a user at the cost of the exclusion of a higher type agent. Hence this condition is a weak notion of fairness in the sense that it is implied by the standard conditions of *free entry*[6] or *envyfreeness*[7]*.* Weak demand monotonicity requires that when the valuation of all agents (weakly) increases for the public good, the set of users should not lose any member. This notion of fairness is implied by *demand monotonicity*[8]. These implications are discussed in Section 5.

[6]See, for example, Deb and Razzolini (1999a, 1999b).
[7]See, for example, Sprumont (2013).
[8]See, for example, Ohseto (2000) and Deb and Razzolini (1999).

Our paper complements the closely related paper of Dobzinski et al. (2008). They show that the equal cost sharing mechanism is a maximal welfare loss minimizer in the class of mechanisms that are strategyproof, budget balanced and that satisfy an axiom called *equal treatment*. The authors write: "An interesting research problem is to characterize the class of mechanisms obtained by dropping the (admittedly strong) equal treatment condition." We do not offer such a characterization because we do not claim that the equal cost sharing mechanism is the unique maximal welfare loss minimizer. However as mentioned earlier, in the two agent case, we show that the equal cost sharing mechanism is a maximal welfare loss minimizer in the larger class of mechanisms when the equal treatment axiom is dropped and budget-balancedness is relaxed to the no-deficit condition.

For more than two agents, the example that we provide shows that the result in Dobzinski et al. (2008) does not hold when budget balancedness is relaxed to the no-budget-deficit condition (but equal treatment is maintained). The Dobzinski et al. result is important because budget balancedness implies a certain notion of efficiency. Surpluses generate wastage when valuable resources are not made use of. If however, surpluses can be committed to other uses or players (not under consideration), then budget balancedness may seem to be a strong restriction. On the other hand, under our fairness criteria, the budget balanced equal cost sharing mechanism is indeed a maximal welfare loss minimizer.

Another related paper is Moulin and Shenker (2001). They consider the provision of a binary, excludable public good when the cost function is a submodular function of the set of users. They show that the mechanism associated with the Shapley value cost sharing formula (which corresponds to the equal cost sharing mechanism for the case of a binary public good with fixed cost of provision) is the unique mechanism that minimizes maximum welfare loss in the class of mechanisms that are defined from a cross monotonic cost sharing method and are group strategyproof, individually rational, non-subsidizing (the cost shares are non negative), budget balanced, and that satisfy consumer sovereignty. Cross monotonicity requires the price paid by a user to weakly decrease when the set of users expands.[9] Group strategyproofness is an incentive compatibility requirement when coalitions of agents are allowed to coordinate messages for mutual benefit. Consumer

[9] Cross monotonicity is related to the axiom of population monotonicity introduced and analyzed in Thomson (1983a, 1983b).

sovereignty ensures that every agent has a valuation that guarantees his participation irrespective of the valuations of other agents. Thus our result applies to a more specialized setting than that in Moulin and Shenker (2001) but establishes the optimality of the equal-cost sharing social choice function amongst a much broader class of mechanisms. We also note that strategyproofness is a more compelling axiom than group strategyproofness from a decision-theoretic perspective. If agents are ignorant of the valuations of other agents, assumptions about the ability of coalitions to coordinate their messages for mutual benefit require stronger justification. Group strategyproofness can also be a demanding requirement in this setting - see Juarez (2008b).

2 Model

An excludable binary public good is to be provided to a *club* of agents. The grand set of agents is $N = \{1, ..., n\}$ and a club is a possibly empty subset of N. If the good is provided to no agent, i.e., if the club is empty, no cost is incurred. Providing the good to any nonempty club generates a cost of 1, regardless of the size or the membership structure of the club.

Each agent $i \in N$ has a type t_i which gives his value for being in a club. We assume that t_i can take on any nonnegative value, in other words, the type space is $T_i = \Re_+$. A type profile is a vector $t \in \Re^n_+$, and is flexibly denoted $t = (t_i, t_{-i})$ for any i, where t_{-i} is a list of the types of all agents except for i. Agents' payoffs are quasilinear in money and there are no informational or allocative externalities. Hence i's payoff is determined by his own type and payment only. In particular it does not depend on who else might be included in the club. To be precise, suppose that club S is formed, i's type is t_i and his payment is p_i. Then his payoff is $I_i(S)t_i - p_i$ where $I_i(S)$ is the indicator function, i.e., $I_i(S) = 1$ if $i \in S$ and $I_i(S) = 0$ otherwise.

A *mechanism* is a function $m = (S^m, p^m) : \Re^n_+ \to 2^N \times \Re^n$, which, for every type profile $t \in \Re^n_+$, determines a (possibly empty) club $S^m(t)$, and a payment $p^m_i(t)$ for every agent $i \in N$. While refering to a particular mechanism $m = (S^m, p^m)$, we will usually drop the superscript m, and write $m = (S, p)$. Furthermore we will feel free to use functions $S(\cdot)$ and $\{p_i(\cdot)\}_{i=1,\dots,n}$ in reference to a given mechanism m whenever no confusion should arise. Note that agents inside or outside the club could be making or receiving payments.

We are interested in a domain of mechanisms which satisfy three basic feasibility requirements: *strategyproofness*, *individual rationality* and *no-budget-deficit*.

Definition 1 A mechanism $m = (S, p)$ is *feasible* if it satisfies the following three conditions.

1. *strategyproofness* (SP): for every $i, t = (t_i, t_{-i})$ and t'_i, $I_i(S(t))t_i - p_i(t) \geq I_i(S(t'_i, t_{-i}))t_i - p_i(t'_i, t_{-i})$.

2. *individual rationality* (IR): for every i and t, $I_i(S(t))t_i - p_i(t) \geq 0$.

3. *no-budget-deficit* (NBD): for every t, $\sum_{i \in N} p_i(t) \geq 1$ whenever $S(t) \neq \varnothing$, and $\sum_{i \in N} p_i(t) \geq 0$ otherwise.

Let Φ be the class of feasible mechanisms. Notice that our feasibility conditions are all ex post, eliminating the need to include a prior on the joint type space in the model. Strategyproofness says that it is a weakly dominant strategy for the agent to report his true type in the revelation game induced by the mechanism. Individual rationality implies that at every type profile each agent receives at least his type-independent payoff from an outside option, which we normalize to zero. Finally no-budget-deficit implies that at every type profile the cost of the club is covered by agents' payments.

We next give without proof a classical result on the characterization of strategyproof and individually rational mechanisms. (See Myerson (1981) for example.)

Proposition 1 A mechanism $m = (S^m, p^m)$ is strategyproof and individually rational if and only if for every $i \in N$, there exist functions $\phi_i^m : \Re_+^{n-1} \to \Re_+ \cup \{\infty\}$ and $h_i^m : \Re_+^{n-1} \to \Re_+$ such that for every $t = (t_i, t_{-i})$

1. if $t_i > \phi_i^m(t_{-i})$, then $i \in S^m(t)$ and $p_i^m(t) = \phi_i^m(t_{-i}) - h_i^m(t_{-i})$,

2. if $t_i < \phi_i^m(t_{-i})$, then $i \notin S^f(t)$ and $p_i^m(t) = -h_i^m(t_{-i})$,

3. if $t_i = \phi_i^m(t_{-i})$, then either $[i \in S^m(t)$ and $p_i^m(t) = \phi_i^m(t_{-i}) - h_i^m(t_{-i})]$ or $[i \notin S^m(t)$ and $p_i^m(t) = -h_i^m(t_{-i})]$.

Thus if m is strategyproof, then whether i will belong to the club at a type profile or not depends on how his type compares to a cutoff $\phi_i^m(t_{-i})$ determined by other agents' types. If his type is above the cutoff, then i belongs to the club. Furthermore his payment is the difference between his cutoff and an amount $h_i^m(t_{-i})$ which, again, is only dependent on others' types. If his type is below the cutoff, then he does not belong to the club and his payment is the negative of $h_i^m(t_{-i})$. If, on the other hand, a mechanism m is generated by functions $\{\phi_i^m, h_i^m\}_{i=1,\ldots,n}$ with these features, then it is strategyproof. Furthermore a strategyproof mechanism m is individually rational if and only if the functions h_i^m are all nonnegative valued.

In this paper, we will always break the tie in case 3 of Proposition 1 in favor of including the agent in the club. None of our results depend on this restriction. Furthermore, whenever convenient, we will drop the superscript m in the functions ϕ_i^m and h_i^m associated with a strategyproof and individually rational mechanism m.

The no-budget-deficit requirement in our notion of feasibility imposes a nontrivial condition on the cutoffs and lump sum amounts associated with a strategyproof and individually rational mechanism. A mechanism m is feasible according to Definition 1 if and only if, at every t,

$$\begin{aligned} S(t) &\neq \varnothing \text{ implies } \sum_{i \in S(t)} \phi_i(t_{-i}) - \sum_{i \in N} h_i(t_{-i}) \geq 1, \text{ and} \\ S(t) &= \varnothing \text{ implies } h_i(t_{-i}) = 0 \text{ for all } i \in N \end{aligned}$$

where the functions $\{\phi_i, h_i\}$ generate m as in Proposition 1. Note that a feasible mechanism necessarily balances the budget whenever it forms the empty club: $p_i(t) = -h_i(t_{-i}) = 0$ for all i. To the best of our knowledge, there is no tractable characterization of the class Φ of feasible mechanisms. We view this as the main difficulty in front of mechanism design in our environment.

Maximal Welfare Loss Efficiency The classical notion of efficiency dictates in our model that a mechanism should either include all in the club, or else, exclude all from the club. More precisely, m is *efficient* if $S(t) = N$ whenever $\sum_{i \in N} t_i \geq 1$, and $S(t) = \varnothing$ otherwise. Hence under efficiency the public good is essentially non-excludable. This stems from the fact that once the club has some member, the marginal net benefit of including another is nonnegative.

Unfortunately this notion of efficiency is incompatible with our feasibility constraints. In other words, there is no $m \in \Phi$ which is efficient. Even though this is a famous result, let us review the argument as it applies in our environment for the special case of two agents.

We will concentrate on three type vectors, $(1,0)$, $(0,1)$ and $(1,1)$. Suppose that m is feasible and efficient. Then $S(1,0) = S(0,1) = S(1,1) = \{1,2\}$ by efficiency. Furthermore $p_2(1,0) = p_1(0,1) = 0$ by IR. Now SP gives $p_2(1,1) = p_1(1,1) = 0$ as well, leading to a budget deficit at $(1,1)$ and hence to a contradition with feasibility.[10]

Given this impossibility, the literature has turned to relaxing feasibility and/or efficiency requirements on mechanisms. Of our interest is a relaxation of efficiency called *maximal welfare loss efficiency*. In order to define this notion, we need some preparations.

First-best welfare at a type profile t, which we will denote $W^*(t)$, is the total welfare in the society achievable under complete information. Hence

$$W^*(t) = \max\left\{\sum\nolimits_{i\in N} t_i - 1, 0\right\}.$$

First-best welfare serves as the benchmark to evaluate the performance of a mechanism under incomplete information. For any $m \in \Phi$ and any type profile t, let $W^m(t)$ be the welfare generated by m at t, i.e.,

$$W^m(t) = \sum\nolimits_{i\in S^m(t)} t_i - \sum\nolimits_{i\in N} p_i^m(t).$$

Now let

$$WL^m(t) = W^*(t) - W^m(t).$$

Thus $WL^m(t)$ is the welfare loss of m at t in comparison to the first-best. Finally let MWL^m denote the *maximal welfare loss* (MWL) of m taken over all type profiles, i.e.,

$$MWL^m = \sup_{t\in\Re^n_+} WL^m(t).$$

For any $m, m' \in \Phi$, we will say that m is MWL superior to m' if $MWL^m < MWL^{m'}$.

[10] According to our definition of efficiency the grand club must be formed at type vectors $(1,0)$ and $(0,1)$, although the empty club would have induced the same total welfare of zero. However the argument would still work if we replaced $(1,0)$ and $(0,1)$ with $(1,\varepsilon)$ and $(\varepsilon,1)$ where ε is a small positive number.

Our mechanism designer will evaluate each mechanism m on the basis of its maximal welfare loss MWL^m and will be interested in employing a mechanism whose maximal welfare loss is minimal.

Definition 2 For any domain $\Phi_0 \subseteq \Phi$, we will say that m^* is *maximal welfare loss efficient in* Φ_0 if

$$m^* \in \arg\min_{m \in \Phi_0} MWL^m.$$

In what follows, we will define and investigate the MWL efficiency properties of a particular mechanism, the equal cost sharing mechanism, which we will denote f.

3 Equal Cost Sharing

We begin by defining the equal cost sharing mechanism, which will be the main focus of our paper. For any set $S \subseteq N$, let $\#S$ denote the cardinality of S.

Definition 3 The *equal cost sharing mechanism* $f = (S, p)$ is defined as follows: for every t

$$\begin{aligned} S(t) &= \bigcup \{S \subseteq N : \textit{for all } i \in S,\ t_i \geq \frac{1}{\#S}\} \\ p_i(t) &= \begin{cases} \frac{1}{\#S(t)} & \text{if } i \in S(t), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly $f \in \Phi$.[11] The mechanism f forms the largest club whose members can individually rationally and equally share cost. Indeed, for every t, $t_i \geq 1/\#S(t)$ for all $i \in S(t)$, in other words, the set $\{S \subseteq N$: for all $i \in S$, $t_i \geq 1/\#S\}$ of clubs is closed under the union operation. To see this, take two clubs T and T' in $\{S \subseteq N$: for all $i \in S$, $t_i \geq 1/\#S\}$. Note that for all $i \in T \cup T'$, either $t_i \geq 1/\#T \geq 1/\#(T \cup T')$ or $t_i \geq 1/\#T' \geq 1/\#(T \cup T')$.

Note that f is in particular *budget-balanced*, i.e., the sum of agents' contributions exactly cover the cost of the club. Furthermore f has the following

[11] Strategyproofness of f follows because the following monotonicity property is satisfied: if $i \in S^f(t_i, t_{-i})$ and $t_i < t'_i$, then $i \in S^f(t'_i, t_{-i})$.

anonymity property: the associated cutoff functions and their arguments are independent of the names of the agents. If there are three agents, for example,

$$\phi_1(x, y) = \phi_2(x, y) = \phi_3(x, y) = \phi_3(y, x).$$

To proceed, we will calculate MWL^f.

Lemma 1 The maximal welfare loss of the equal cost sharing mechanism f is $\sum_{k=2}^{n} 1/k$.

Proof. We will first show that $WL^f(t) < \sum_{k=2}^{n} 1/k$. If $\sum_{i \in N} t_i < 1$, then $W^*(t) = W^f(t) = 0$ giving $WL^f(t) = 0$ as well. So take any t such that $\sum_{i \in N} t_i \geq 1$ so that $W^*(t) = \sum_{i \in N} t_i - 1$. There are two cases. If $S^f(t) = N$, then $W^f(t) = \sum_{i \in N} t_i - 1 = W^*(t)$ and $WL^f(t) = 0$. If $\#S^f(t) = n_0 \in \{0, 1, ..., n-1\}$, then let $(t_{(1)}, t_{(2)}, ..., t_{(n-n_0)})$ be a listing of the types of agents in $N \backslash S^f(t)$ from highest to lowest, with ties broken arbitrarily. We claim that $t_{(k)} < \frac{1}{n_0+k}$ for every $k = 1, ..., n - n_0$. If not, let k^* be the largest $k \in \{1, ..., n - n_0\}$ such that $t_{(k)} \geq \frac{1}{n_0+k}$. It follows that $t_{(k)} \geq \frac{1}{n_0+k^*}$ for all $k \leq k^*$ and $S^f(t)$ should have had $n_0 + k^*$ instead of n_0 members, a contradiction. Hence

$$WL^f(t) = \sum_{i \notin S^f(t)} t_i = \sum_{k=1}^{n-n_0} t_{(k)} < \sum_{k=1}^{n-n_0} 1/(n_0 + k) \leq \sum_{k=2}^{n} 1/k$$

giving $WL^f(t) < \sum_{k=2}^{n} 1/k$ as we wanted to show.

Hence for all t, $WL^f(t) < \sum_{k=2}^{n} 1/k$ giving $MWL^f \leq \sum_{k=2}^{n} 1/k$. To finish, we establish the reverse inequality by taking a sequence of type vectors at which f generates welfare losses converging to $\sum_{k=2}^{n} 1/k$. For all sufficiently small $\varepsilon > 0$, let $t^\varepsilon = (1 - \varepsilon, \frac{1}{2} - \varepsilon, ..., \frac{1}{n} - \varepsilon)$. Note $S^f(t^\varepsilon) = \varnothing$ and if $\sum_{i \in N} t_i^\varepsilon \geq 1$, which happens if ε is small enough, then $WL^f(t^\varepsilon) = \sum_{i \in N} t_i^\varepsilon - 1 = \sum_{k=2}^{n} 1/k - n\varepsilon$. Now as $\varepsilon \downarrow 0$, $WL^f(t^\varepsilon) \to \sum_{k=2}^{n} 1/k$, indicating that $MWL^f = \sup_t WL^f(t) \geq \sum_{k=2}^{n} 1/k$. ■

We will now show that when there are only two agents, there is no feasible mechanism in Φ whose MWL is less than that of f.

Proposition 2 The equal cost sharing mechanism f is maximal welfare loss efficient in Φ if $N = \{1, 2\}$.

Proof. By Lemma 1, $MWL^f = 1/2$. Suppose $MWL^g < 1/2$ for some $g \in \Phi$. We first claim that for all sufficiently small $\varepsilon > 0$, we must have $S^g(1-\varepsilon, \frac{1}{2}-\varepsilon) \neq \varnothing$. Otherwise $W^g(1-\varepsilon, \frac{1}{2}-\varepsilon) = 0$ and $WL^g(1-\varepsilon, \frac{1}{2}-\varepsilon) = \frac{1}{2} - 2\varepsilon$ giving $MWL^g \geq \lim_{\varepsilon \downarrow 0} WL^g(1-\varepsilon, \frac{1}{2}-\varepsilon) = 1/2$, a contradiction. Similarly we must also have $S^g(\frac{1}{2}-\varepsilon, 1-\varepsilon) \neq \varnothing$ for all sufficiently small $\varepsilon > 0$. Fix such ε. It follows that $S^g(1-\varepsilon, \frac{1}{2}-\varepsilon) = S^g(\frac{1}{2}-\varepsilon, 1-\varepsilon) = \{1,2\}$ as neither individual alone could individually rationally cover the cost of a nonempty club at these type profiles. Now SP and IR imply

$$\begin{aligned}
S^g(1-\varepsilon, 1-\varepsilon) &= \{1,2\} \\
p_1^g(1-\varepsilon, 1-\varepsilon) &= p_1^g(\frac{1}{2}-\varepsilon, 1-\varepsilon) \leq \frac{1}{2} - \varepsilon \text{ and} \\
p_2^g(1-\varepsilon, 1-\varepsilon) &= p_2^g(1-\varepsilon, \frac{1}{2}-\varepsilon) \leq \frac{1}{2} - \varepsilon,
\end{aligned}$$

leading to a budget deficit at the type vector $(1-\varepsilon, 1-\varepsilon)$. Hence g could not have been feasible. ■

It turns out, however, that when N contains three agents or more, f is no longer maximal welfare loss efficient in Φ. To establish this, we will later present an example of a feasible mechanism which is MWL superior to f. We will first show, however, that if certain notions of fairness are imposed in the definition of feasibility, then f regains MWL optimality. In other words, we will show that there exist subsets Φ generated by fairness considerations, where f is MWL efficient even with $n > 2$ agents. To this end, we introduce two distinct and arguably rather weak notions of fairness.

Definition 4 A mechanism $m = (S, p)$ satisfies *weak demand monotonicity* (wDM) if $S(t) \subseteq S(t')$ whenever $t_i \leq t'_i$ for all $i \in N$.

Definition 5 A mechanism $m = (S, p)$ satisfies *weak free entry* (wFE) if for every i, j and t, $i \in S(t)$ whenever $j \in S(t)$ and $t_i > t_j$.

Note that neither condition has an imposition on how payments are determined. In a discussion section to follow, we will argue that these two conditions are independent and that they are weaker than the demand monotonicity, free entry and envy-freeness conditions that appear in the literature. Let Φ_{wDM} and Φ_{wFE} denote the classes of feasible mechanisms satisfying wDM and wFE respectively. It is clear that $f \in \Phi_{wDM} \cap \Phi_{wFE}$. Imposing either wDM or wFE is sufficient to obtain the MWL efficiency of f.

Proposition 3 *The equal cost sharing mechanism f is maximal welfare loss efficient in $\Phi_{wDM} \cup \Phi_{wFE}$.*

Proof. We will prove this result here for the special case when $N = \{1, 2, 3\}$. The argument extends to $n > 3$ agents at the cost of some investment in notation as we show in Appendix 1.

By Lemma 1, $MWL^f = \frac{5}{6}$. Suppose that $MWL^g < \frac{5}{6}$ for some $g \in \Phi_{wDM} \cup \Phi_{wFE}$. Our proof will rely on how g necessarily behaves at type profiles

$$(1 - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{3} - \varepsilon), \ (1 - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon), \text{ and } (1 - \varepsilon, 1 - \varepsilon, 1 - \varepsilon)$$

and their permutations for small enough but strictly positive ε. We recall from Lemma 1 that MWL^fobtains at all 6 permutations of the profile $(1 - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{3} - \varepsilon)$ as ε falls to 0.

We first claim that for $\varepsilon > 0$ small enough, g must form a nonempty club at the type profiles $(1 - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{3} - \varepsilon)$ and $(\frac{1}{2} - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon)$. If not, the welfare loss of g at these profiles would be $\frac{5}{6} - 3\varepsilon$, and taking the limit as ε falls to zero would give $MWL^g = \frac{5}{6}$, a contradiction. Now fix a sufficiently small $\varepsilon > 0$ such that $S^g(1 - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{3} - \varepsilon)$ and $S^g(\frac{1}{2} - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon)$ are both nonempty.

We now claim that

$$3 \in S^g(1 - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon).$$

There are two cases to consider. First suppose that $g \in \Phi_{wDM}$. If $3 \in S^g(1-\varepsilon, \frac{1}{2}-\varepsilon, \frac{1}{3}-\varepsilon)$ or $3 \in S^g(\frac{1}{2}-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon)$, then $3 \in S^g(1-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon)$ by wDM. If not, then

$$S^g(1 - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{3} - \varepsilon) = \{1, 2\} = S^g(\frac{1}{2} - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon)$$

by feasibility. Now SP and IR give

$$\begin{aligned}
\{1, 2\} &\subseteq S^g(1 - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon), \\
p_1^g(1 - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon) &= p_1^g(\frac{1}{2} - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon) < \frac{1}{2}, \text{ and} \\
p_2^g(1 - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon) &= p_2^g(1 - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{3} - \varepsilon) < \frac{1}{2}.
\end{aligned}$$

Hence $3 \in S^g(1-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon)$ so that no-budget-deficit obtains.

Suppose now that $g \in \Phi_{wFE}$. Then $1 \in S^g(\frac{1}{2}-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon)$ since otherwise $S^g(\frac{1}{2}-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon) = \{2,3\}$ and wFE fails. Similarly $2 \in S^g(1-\varepsilon, \frac{1}{2}-\varepsilon, \frac{1}{3}-\varepsilon)$. Consequently

$$\begin{aligned}
\{1,2\} &\subseteq S^g(1-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon), \\
p_1^g(1-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon) &= p_1^g(\frac{1}{2}-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon) < \frac{1}{2}, \text{ and} \\
p_2^g(1-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon) &< p_2^g(1-\varepsilon, \frac{1}{2}-\varepsilon, \frac{1}{3}-\varepsilon) < \frac{1}{2}.
\end{aligned}$$

Thus we must have, once again, $3 \in S^g(1-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon)$ to clear the budget.

The exact same reasoning applies in showing that

$$1 \in S^g(\frac{1}{3}-\varepsilon, 1-\varepsilon, 1-\varepsilon) \text{ and } 2 \in S^g(1-\varepsilon, \frac{1}{3}-\varepsilon, 1-\varepsilon).$$

Thus $S^g(1-\varepsilon, 1-\varepsilon, 1-\varepsilon) = \{1,2,3\}$ by SP. However the payments at this type profile do not cover the cost of the club. For example $p_1^g(1-\varepsilon, 1-\varepsilon, 1-\varepsilon) = p_1^g(\frac{1}{3}-\varepsilon, 1-\varepsilon, 1-\varepsilon) < \frac{1}{3}$. Similarly $p_2^g(1-\varepsilon, 1-\varepsilon, 1-\varepsilon)$ and $p_3^g(1-\varepsilon, 1-\varepsilon, 1-\varepsilon)$ are both strictly less than $\frac{1}{3}$. Hence g could not have been feasible. ■

It is instructive to juxtapose the proofs of Propositions 2 and 3 in order to understand the complications that arise with three or more agents. Both proofs rely on exploiting the behavior of mechanisms at hand at certain *critical* type profiles. For small enough $\varepsilon > 0$, take the profile $(1-\varepsilon, \frac{1}{2}-\varepsilon)$ in the two-agent case, and the profile $(1-\varepsilon, \frac{1}{2}-\varepsilon, \frac{1}{3}-\varepsilon)$ in the three-agent case. To beat f in MWL, an alternate mechanism must form nonempty clubs at these profiles. Now in the former scenario, this nonempty club is necessarily the grand coalition $\{1,2\}$. In the latter, however, the nonempty club is either the grand coalition $\{1,2,3\}$, or one of the doubletons $\{1,2\}$ and $\{1,3\}$. It is precisely this multiplicity of possible clubs at critical profiles that creates the complications with three or more agents. We need, in order to attain a budget deficit at the profile $(1-\varepsilon, 1-\varepsilon, 1-\varepsilon)$, that at every permutation of the profile $(1-\varepsilon, 1-\varepsilon, \frac{1}{3}-\varepsilon)$ the agent with the lowest type $\frac{1}{3}-\varepsilon$ to belong to the club formed. Unfortunately feasibility of a mechanism alone may fail to lead to this conclusion as our Example 1 below shows. As a result we need wFE or wDM for the argument to work.

In order to show that Proposition 3 is tight, in other words, to show that a failure of wDM *and* wFE leads to the suboptimality of f, we will next present a class of feasible mechanisms obtained from small perturbations of f. In doing so we will make use of the anonymity property, which is also satisfied by the perturbations by construction.

Example 1 Suppose $N = \{1, 2, 3\}$ and take $\varepsilon \in (0, \frac{1}{24})$. Consider the following perturbation of the equal cost sharing mechanism f, which we call f^{ε} and present together with f for comparison purposes. Both f anf f^{ε} are anonymous and the diagrams give the cutoffs of some $k \in N$ as a function of (t_i, t_j) where i, j and k are distinct agents in N.

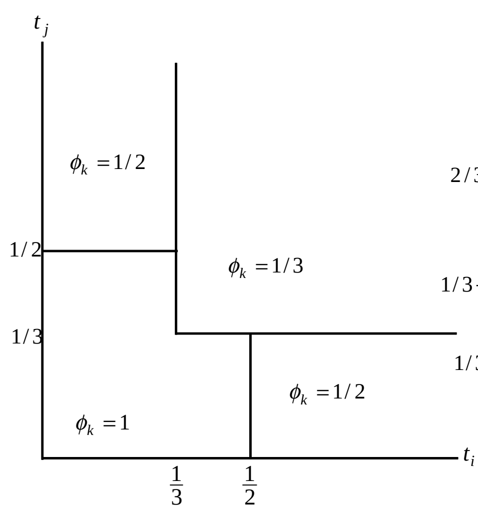


FIGURE 1: f when $n=3$

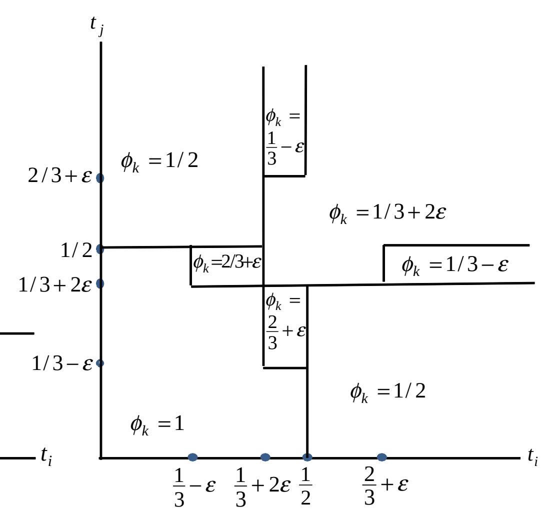


FIGURE 2: f^{ε} when $n=3$

In both diagrams the arguments of ϕ_k are omitted to economize on space. For both mechanisms, at any vector (t_i, t_j) which lies at the border of two or more distinct regions, the effective cutoff is the smallest one. At $(t_i, t_j) = (\frac{1}{3}, \frac{1}{2})$ for example, f imposes a cutoff of $\frac{1}{3}$ on agent k. Similarly at $(t_i, t_j) = (\frac{2}{3} + \varepsilon, \frac{1}{2})$, f^{ε} imposes the cutoff $\frac{1}{3} - \varepsilon$ on k. Both mechanisms charge each member of any nonempty club exactly his cutoff. Hence their h_i functions are zero-valued.

We would like to point out various features of the mechanism f^{ε} in Example 1.

1. Generated by symmetric cutoff functions as given in the plot, f^ε satisfies SP. Furthermore since the associated h_i functions are zero-valued, IR obtains and members of any club pay their cutoffs. In Appendix 2, we exhibit how f^ε behaves in detail by partitioning $\Re^3_+$ into 6 distinct sets of type vectors and analyzing them individually.

2. As ε vanishes, the mechanism f^ε converges (pointwise) to f. Hence we interpret f^ε to be a perturbation of f.

3. In order to calculate MWL^{f^ε}, take the type vectors $(\frac{1}{3} - \varepsilon, \frac{1}{2}, 1)$ and $(\frac{1}{3} + 2\varepsilon, \frac{1}{3} + 2\varepsilon, 1)$ or any one of their permutations. At type vectors sufficiently close to these vectors from below, we will now show that f^ε forms the empty club. Let us verify this leisurely. First take $(\frac{1}{3}-\varepsilon, \frac{1}{2}, 1)$ and a sufficiently small $\delta > 0$. Omitting subscripts note that the cutoffs imposed by f^ε are
$$\begin{aligned} \phi(\frac{1}{3} - \varepsilon - \delta, \frac{1}{2} - \delta) &= 1, \\ \phi(\frac{1}{3} - \varepsilon - \delta, 1 - \delta) &= \frac{1}{2}, \text{ and} \\ \phi(\frac{1}{2} - \delta, 1 - \delta) &= \frac{1}{3} - \varepsilon \end{aligned}$$
Hence if t is sufficiently close to $(\frac{1}{3}-\varepsilon, \frac{1}{2}, 1)$ from below, then $S(t) = \varnothing$, causing a welfare loss of $\frac{5}{6} - \varepsilon$.
Simiarly If we take $(\frac{1}{3} + 2\varepsilon, \frac{1}{3} + 2\varepsilon, 1)$ and a small $\delta > 0$, we get
$$\begin{aligned} \phi(\frac{1}{3} + 2\varepsilon - \delta, \frac{1}{3} + 2\varepsilon - \delta) &= 1, \\ \phi(\frac{1}{3} + 2\varepsilon - \delta, 1 - \delta) &= \frac{1}{2}. \end{aligned}$$
Note that as $\varepsilon < \frac{1}{24}$, $\frac{1}{3}+2\varepsilon < \frac{1}{2}$. Hence at type vectors sufficiently close to $(\frac{1}{3}+2\varepsilon, \frac{1}{3}+2\varepsilon, 1)$ from below, no agent meets his cutoff and the empty club is formed. The associated welfare loss this time is $\frac{2}{3} + 4\varepsilon < \frac{5}{6}$ as $\varepsilon < \frac{1}{24}$. In either case $MWL^{f^\varepsilon} < MWL^f$. Appendix 2 demonstrates in detail the calculation of MWL^{f^ε}.

4. An interesting question is to identify that ε for which MWL^{f^ε} is the lowest among $\{f^\varepsilon : 0 < \varepsilon < \frac{1}{24}\}$. This obtains by equating the two

candidates for MWL^{f^ε} above: $\frac{5}{6} - \varepsilon = \frac{2}{3} + 4\varepsilon$, which gives $\varepsilon = \frac{1}{30}$. In other words

$$\inf_{0<\varepsilon<\frac{1}{24}} MWL^{f^\varepsilon} = MWL^{f^{1/30}} = \frac{4}{5}.$$

5. Our perturbation f^ε fails both wDM and wFE. For example, f^ε forms the club $\{2,3\}$ at the type profile $(\frac{1}{3} + 2\varepsilon, \frac{2}{3} + \varepsilon, \frac{1}{3})$. However when all types are larger at $(1, 1, \frac{1}{3} + \varepsilon)$ f^ε excludes 3 and forms the club $\{1,2\}$. This is a failure of wDM. Furthermore, the fact that agent 1 does not belong to the club $\{2,3\}$ at $(\frac{1}{3} + 2\varepsilon, \frac{2}{3} + \varepsilon, \frac{1}{3})$ even though $t_1 > t_3$ is a failure of wFE. This was of course to be expected because of Proposition 3 above. Any mechanism which satisfies wDM or wFE could not possibly be superior to f in MWL.

6. We would like to re-emphasize that f^ε is by construction *anonymous*: for every i, t and permutation π on $\{1,2,3\}$,

$$f_i^\varepsilon(t_1, t_2, t_3) = f_{\pi(i)}^\varepsilon(t_{\pi(1)}, t_{\pi(2)}, t_{\pi(3)}).$$

   Let Φ_A be the class of anonymous and feasible mechanisms. Thus the equal cost sharing mechanism f is not MWL efficient in Φ_A as $f^\varepsilon \in \Phi_A$. This notion of anonymity is, of course, stronger than anonymity in utility terms, which appears in Sprumont (2013). It appears in Dobzinski et al. (2008) where it is called equal treatment.

7. As opposed to the equal sharing mechanism f, its perturbation f^ε is not budget balanced. However the budget surplus induced by f^ε is at most 6ε and this is not large enough to offset the efficiency gain in the worst-case scenario. Details are in Appendix 2.

4 Discussion

To the best of our knowledge, wDM and wFE are novel modifications of closely related (and stronger) conditions that appear in the literature. Note that there is no logical implication between our conditions. To see this suppose $N = \{1,2\}$. Suppose that $S(t) = \{1\}$ if $t_1 \geq 1$ and $S(t) = \varnothing$ otherwise, with agent 1 being charged the full cost of the club whenever he is in the club. This mechanism is in Φ, it satisfies wDM and fails wFE. On the other hand the mechanism given by $S(t) = \{i\}$ if $t_j \leq 1 \leq t_i$ and $S(t) = \varnothing$ otherwise, with the agent in the club covering the cost of the club fully belongs to Φ, satisfies wFE but fails wDM. Thus the class $\Phi_{wDM} \cup \Phi_{wFE}$ on which f^e is MWL efficient is a strict superset of both Φ_{wDM} and Φ_{wFE}.

We note that Proposition 2 is not a corollary to Proposition 3, as there exist feasible mechanisms outside $\Phi_{wDM} \cup \Phi_{wFE}$ in 2-agent environments. As an example, consider the mechanism defined by $S(t) = \varnothing$ if $t_1 < 1$, $S(t) = \{1, 2\}$, $p_1(t) = 1$ and $p_2(t) = 0$ if $1 \leq t_1 < 2$ and $S(t) = \{1\}$ and $p_1(t) = 1$ if $t_1 \geq 2$.

Our wDM condition is a straightforward weakening of the *demand monotonicity* condition that appears in Ohseto (2000), which requires, on top of our wDM, that $S(t) = S(t')$ whenever $t_i \leq t'_i$ for every $i \in S(t)$ and $t'_i \leq t_i$ for every $i \notin S(t)$.

Furthermore for any feasible mechanism, our wFE condition is implied by the free entry condition which appears in Deb and Razzolini (1999). A mechanism m satisfies *free entry* if for every i, j and t, $i \in S(t)$ whenever $j \in S(t)$ and $t_i > p_j(t)$. Suppose that m is a feasible mechanism which satisfies free entry. If for some t, i and j, $j \in S^m(t)$ and $t_i > t_j$, then $t_i > p_j^m(t)$ as well by individual rationality. Consequently $i \in S^m(t)$ by free entry. This establishes that m satisfies wFE as well.

A different condition that implies wFE is the classical notion of envy-freeness (see, for example, Sprumont 2013). A mechanism m is *envy-free* if for every t, i and j, $I_i(S(t))t_i - p_i(t) \geq I_j(S(t))t_i - p_j(t)$. Suppose that m is a feasible mechanism. If at some type profile t, $j \in S(t)$ and $i \notin S(t)$ even though $t_i > t_j$, then i envies j since

$$\begin{aligned} I_i(S(t))t_i - p_i(t) &= 0 \\ &< t_i - t_j \\ &\leq I_j(S(t))t_i - p_j(t) \end{aligned}$$

where the weak inequality follows by individual rationality. Hence if m is an envy-free and feasible mechanism, it also satisfies wFE.

5 Conclusion

Equal cost sharing mechanism is a simple and appealing procedure with desirable incentive properties. We show in this paper that, in general, it is not maximal welfare loss efficient. Hence a mechanism designer with the worst-case-scenario in mind, may opt to employ a different mechanism, perhaps a small perturbation of equal cost sharing which may lead to budget surplus as we exhibit in Example 1. If the designer is restricted to use a mechanism with certain fairness properties (as embodied in our weak demand monotonicity or weak free entry conditions), however, then equal cost sharing can not be improved upon.

We leave for future work the investigation of the consequences of replacing the no-budget-deficit condition in our feasibility definition with strict budget balance. As we remarked above, our perturbation of the equal cost sharing mechanism which fares better in terms of maximal welfare loss leads to budget surpluses. Hence it may very well be that equal cost sharing mechanism is maximal welfare loss efficient within the class of feasible and budget balanced mechanisms.

We also leave for future work the ambitious question of how to solve the mechanism design problem

$$\min_{m\in\Phi} MWL^m.$$

We perceive the main difficulty here to be the absence of a tractable characterization of the set Φ of strategyproof, individually rational and no-budget-deficit mechanisms.

References

[1] Deb, R. and Razzolini, L. "Auction-like mechanisms for pricing excludable public goods," Journal Economic Theory 88, 340-368 (1999a).

[2] Deb, R. and Razzolini, L. "Voluntary cost sharing for an excludable public project," Mathematical Social Sciences 37, 123-138 (1999b).

[3] Dobzinski, S., Mehta, A., Roughgarden, T., and Sundararajan, M. "Is Shapley cost sharing optimal?" In SAGT'08 Proceedings of the 1st International Symposium on Algorithmic Game Theory (2008).

[4] Juarez, R. "The worst absolute loss in the problem of the commons: random priority versus average cost," Economic Theory 34, 69-84 (2008a).

[5] Juarez, R. "Optimal group strategyproof cost sharing: budget-balance vs. efficiency," mimeo. University of Hawaii (2008b).

[6] Juarez, R. "Group strategy proof of cost sharing: mechanisms without indi§erences," mimeo (2008c).

[7] Koutsoupias, E. and Papadimitriou, C. "Worst-case equilibria." In the 16th Annual Symposium on Theoretical Aspects of Computer Science 404-413 (1999).

[8] Mehta, A., Roughgarden, T., and Sundararajan, M. "Beyond Moulin mechanisms." In Proceedings of the 8th ACM Conference on Electronic Commerce (EC), 1-10 (2007).

[9] Moulin H. "The price of anarchy of serial, average and incremental cost sharing," Economic Theory 36, 379-405 (2008).

[10] Moulin, H. and Shenker, S. "Strategyproof sharing of submodular costs: budget balance versus e¢ ciency," Economic Theory 18, 511-533 (2001).

[11] Mutuswami, S. "Strategyproofness, non-bosiness, and group strategyproofness in a cost model," Economics Letters 89, 83-88 (2005).

[12] Mutuswami, S. "Strategyproof cost sharing of a binary good and the egalitarian solution," Mathematical Social Sciences 48, 271-280 (2008).

[13] Myerson, R. "Optimal auction design," Mathematics of Operations Research 6, 58-73 (1981).

[14] Nisan, N. "Introduction to mechanism design (for computer scientists)." In Algorithmic Game Theory, edited by N. Nisan, T. Roughgarden, E. Tardos, and V.V. Vazirani. Cambridge University Press, New York (2008).

[15] Ohseto, S. "Characterizations of strategy-proof mechanisms for excludable versus nonexcludable public p rojects," Games and Economic Behavior 32, 51-66 (2000).

[16] Ohseto, S. "Augmented serial rules for an excludable public good," Economic Theory 26, 589-606 (2005).

[17] Olszewski, W. "Coalition strategy-proof mechanisms for provision of excludable public goods," Games and Economic Behavior 46, 88-114 (2004).

[18] Roughgarden, T. "The price of anarchy is independent of the network topology," Symposium on Theory of Computing 428-437 (2002).

[19] Roughgarden, T. and Sundararajan, M. "New trade-offs in cost-sharing mechanisms." In Proceedings of the 38th Annual ACM Symposium on the Theory of Computing (STOC) 79-88 (2006).

[20] Roughgarden, T. and Tardos, E. "How bad is selfish routing?," Journal of the Association for Computing Machinery 49, 236-259 (2002).

[21] Sprumont, Y. "Population monotonic allocation schemes for cooperative games with transferable utility," Games and Economic Behavior 2, 378-394 (1990).

[22] Sprumont, Y. "Constrained-optimal Strategy-proof Assignment: Beyond the Groves Mechanisms," Journal of Economic Theory (2013), http://dx.doi.org/10.1016/j.jet.2012.09.015.

[23] Thomson, W. "The fair division of a fixed supply among a growing population," Mathematics of Operations Research 8, 319-326 (1983a).

[24] Thomson, W. "Problems of fair division and the egalitarian principle," Journal of Economic Theory 31, 211-226 (1983b).

[25] Yu, Y. "Serial cost sharing of an excludable public good available in multiple units," Social Choice and Welfare 29, 539-555 (2007).

Appendix 1: Proof of Proposition 3

Let $N = \{1, ..., n\}$ where $n > 3$. Here we will generalize the argument presented in the main text for $n = 3$. To this end we will need to develop some notation. For any type vector $t \in \Re^n_+$, let $P(t)$ be the set of all permutations of t. In other words $t' \in P(t)$ iff there is a bijection $\pi : N \to N$ such that for every i, $t_i = t'_{\pi(i)}$. Now for any $\varepsilon \in (0, \frac{1}{n})$ and any $k = 1, ..., n$, define the type vector

$$t_i^k(\varepsilon) = \begin{cases} 1 - \varepsilon & \text{if } i \leq k \\ \frac{1}{i} - \varepsilon & \text{if } k < i \end{cases}$$

Hence

$$\begin{aligned} t^1(\varepsilon) &= (1 - \varepsilon, \frac{1}{2} - \varepsilon, \frac{1}{3} - \varepsilon, ..., \frac{1}{n} - \varepsilon), \\ t^2(\varepsilon) &= (1 - \varepsilon, 1 - \varepsilon, \frac{1}{3} - \varepsilon, ..., \frac{1}{n} - \varepsilon), \\ &\cdots \\ t^n(\varepsilon) &= (1 - \varepsilon, ..., 1 - \varepsilon). \end{aligned}$$

Now let

$$T^k(\varepsilon) = P(t^k(\varepsilon)) \text{ for every } 1, ..., k.$$

Note that k also indicates the number of agents whose types are $1 - \varepsilon$ in every element of $T^k(\varepsilon)$. Furthermore if $k > 1$, then for every $t \in T^k(\varepsilon)$, there exists some $t' \in T^{k-1}(\varepsilon)$ such that

$$t_i = \begin{cases} t'_i & \text{if } t'_i \neq \frac{1}{k} - \varepsilon, \text{ and} \\ 1 - \varepsilon & \text{if } t'_i = \frac{1}{k} - \varepsilon. \end{cases}$$

Hence for every $k > 1$ and every $t \in T^k(\varepsilon)$, there exist k vectors $t^1, ..., t^k \in T^{k-1}(\varepsilon)$ such that t is obtained by increasing the type of the agent with $\frac{1}{k} - \varepsilon$ in any one of $t^1, ..., t^k$ to $1 - \varepsilon$.

If ε is small enough, any mechanism which has a lower maximal welfare loss than equal cost sharing must form a nonempty club at all members of $T^k(\varepsilon)$ for every k. We record this rather straightforward observation next.

Lemma 2 *If $MWL^g < MWL^f$, then there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ and $t \in \bigcup_{k=1}^n T^k(\varepsilon)$, $S^g(t) \neq \varnothing$.*

Proof. Take a mechanism g with $MWL^g < MWL^f$. Recall that $MWL^f = \sum_{i=2}^n 1/i$. By construction of the sets $\{T^k(\varepsilon)\}_{k=1,...,n}$, if $t \in T^{k-1}(\varepsilon)$ and $t' \in T^k(\varepsilon)$, then $\sum_{i=1}^n (t'_i - t_i) = 1 - \frac{1}{k} > 0$ for $k > 2$. Consequently for any list $\{t^k\}_{k=1,...,n}$ where $t^k \in T^k(\varepsilon)$ for every k, the number $\sum_{i=1}^n t_i^k$ is increasing in k. Thus if $S^g(t^k) = \varnothing$ for all k, then $WL^g(t^n) > \cdots > WL^g(t^1)$. So it suffices to show that $S^g(t) \neq \varnothing$ for every $t \in T^1(\varepsilon)$. Suppose, towards a contradiction, that for every $\bar{\varepsilon} > 0$, there exists some $\varepsilon \in (0, \bar{\varepsilon})$ and $t(\varepsilon) \in T^1(\varepsilon)$ such that $S^g(t(\varepsilon)) = \varnothing$. Then there exists a sequence $\{\varepsilon^q\} \searrow 0$ and a corresponding sequence of type vectors $t^q \in T^1(\varepsilon^q)$ such that $S^g(t^q) = \varnothing$ for all q. Note that such t^q is a permutation of the type vector $(1-\varepsilon, \frac{1}{2} - \varepsilon, ..., \frac{1}{n} - \varepsilon)$. Then $WL^g(t^q) \to \sum_{i=2}^n 1/i$, a contradiction to the hypothesis $MWL^g < \sum_{i=2}^n 1/i$. Thus our supposition is false: there exists $\bar{\varepsilon} > 0$ such that if $\varepsilon \in (0, \bar{\varepsilon})$ and $t \in T^1(\varepsilon)$, $S^g(t) \neq \varnothing$. ■

From now on we will fix such small $\varepsilon \in (0, \bar{\varepsilon})$ and to economize on notation we will write T^k instead of $T^k(\varepsilon)$. We will also denote by t^* the type vector $(1-\varepsilon, ..., 1-\varepsilon)$, the unique element of T^n. Our goal is to show that if (1) g is strategyproof and individually rational, (2) $MWL^g < \sum_{k=2}^n \frac{1}{k}$ and (3) g satisfies wDM *or* wFE, then g runs a budget deficit at some $t \in \bigcup_{k=1}^n T^k$.

To begin, take a strategyproof and individually rational mechanism g such that $MWL^g < \sum_{k=2}^n \frac{1}{k}$.

Case 1: Weak Demand Monotonicity Suppose that g additionally satisfies wDM. Write

$$T^{n-1} = \{t^{1,n-1}, t^{2,n-1}, ..., t^{n,n-1}\},$$

where for every k, $t_k^{k,n-1} = \frac{1}{n} - \varepsilon$ and $t_i^{k,n-1} = 1 - \varepsilon$ for all $i \neq k$. Note that the only difference between the type profiles $t^{k,n-1}$ and t^* is agent k's type:

$$t_i^* = \begin{cases} t_i^{k,n-1} & \text{if } i \neq k \text{, and} \\ 1 - \varepsilon > t_k^{k,n-1} & \text{if } i = k. \end{cases}$$

This observation has an important consequence, which we will use recursively below. Note that $S^g(t^{k,n-1}) \neq \varnothing$ by the Lemma above. Now if $k \in S^g(t^{k,n-1})$, then, by SP and IR,

$$k \in S^g(t^*) \text{ and } p_k(t^*) = p_k(t^{k,n-1}) \leq \frac{1}{n} - \varepsilon.$$

There are two possibilities: (1) $k \in S^g(t^{k,n-1})$ for every k and g runs a budget deficit at t^*, in which case the proof for Case 1 is complete, or (2) $k' \notin S^g(t^{k',n-1})$ for some k'. Suppose the latter case, and in particular, without loss of generality, $k' = 1$.

Now consider the following subset of T^{n-2}:

$$T^{n-2}(1) = \{t \in T^{n-2} : t_1 = \frac{1}{n} - \varepsilon\}.$$

Hence $t \in T^{n-2}(1)$ iff $n-2$ agents have type $1-\varepsilon$, agent 1 has type $\frac{1}{n} - \varepsilon$ and a *different* agent has type $\frac{1}{n-1} - \varepsilon$. Now to pin down this *different* agent, we write

$$T^{n-2}(1) = \{t^{2,n-2}, t^{3,n-2}, ..., t^{n,n-2}\}.$$

where $t_k^{k,n-2} = \frac{1}{n-1} - \varepsilon$. Note that for every $k > 1$, $t_1^{1,n-1} = t_1^{k,n-2} = \frac{1}{n} - \varepsilon$ and

$$t_i^{1,n-1} = \begin{cases} t_i^{k,n-2} & \text{if } i \neq k \text{, and} \\ 1 - \varepsilon > t_k^{k,n-2} & \text{if } i = k. \end{cases}$$

Note that since $t_i^{1,n-1} \leq t_i^{k,n-2}$ for all i and since we supposed $1 \notin S^g(t^{1,n-1})$ above, wDM implies $1 \notin S^g(t^{k,n-2})$ for any $k = 2, ..., n$. However $S^g(t^{k,n-2}) \neq \varnothing$ by the Lemma. Now using the exact same logic of the previous paragraph, if some $k \in S^g(t^{k,n-2})$, then, by SP and IR,

$$k \in S^g(t^{1,n-1}) \text{ and } p_k(t^{1,n-1}) = p_k(t^{k,n-2}) \leq \frac{1}{n-1} - \varepsilon.$$

Once again, there are two possibilities: (1) $k \in S^g(t^{1,n-1})$ for all $k > 1$, leading to a budget deficit and the termination of the proof of Case 1, or (2) $k' \notin S^g(t^{k',n-2})$ for some $k' > 1$. Suppose the latter case and, without loss of generality, $k' = 2$.

Continuing in this fashion, suppose $k \notin S^g(t^{k,n-k})$ for all $k = 1, ..., n-2$ and write

$$T^1(1, 2, ..., n-2) = \{t^{n-1,1}, t^{n,1}\}.$$

Now wDM implies that agents $1, ..., n-2$ should belong to neither $S^g(t^{n-1,1})$ nor $S^g(t^{n,1})$. However, since a club must be formed at both type vectors, we must have

$$S^g(t^{n-1,1}) = S^g(t^{n,1}) = \{n-1, n\}.$$

This follows because $t^{n-1,1}_{n-1} = t^{n,1}_{n} = \frac{1}{2} - \varepsilon$ and $t^{n-1,1}_{n} = t^{n,1}_{n-1} = 1 - \varepsilon$. Thus at either type vector no agent can individually rationally finance the club alone. Hence, by SP and IR,

$$\begin{aligned} n-1, n &\in S^g(t^{n-2,2}), \\ p^g_{n-1}(t^{n-1,1}) &= p^g_{n-1}(t^{n-2,2}) \leq \frac{1}{2} - \varepsilon, \text{ and} \\ p^g_{n}(t^{n,1}) &= p^g_{n}(t^{n-2,2}) \leq \frac{1}{2} - \varepsilon. \end{aligned}$$

But now to avoid a budget deficit at $t^{n-2,2}$ agent $n-2$ must be included in $S^g(t^{n-2,2})$ which is the contradiction we need to finish the proof of Case 1.

Case 2: Weak Free Entry Now suppose that g additionally satisfies wFE . Define for every $k = 1, ..., n-1$ and every $t \in T^k$

$$\begin{aligned} N^k_k(t) &= \{i \in N : t_i = 1 - \varepsilon\} \text{ and} \\ N^k_{k+1}(t) &= \{i \in N : t_i \geq \frac{1}{k+1} - \varepsilon\}. \end{aligned}$$

Note that $N^k_k(t)$ and $N^k_{k+1}(t)$ differ by a unique element, and this is the agent whose type is $\frac{1}{k+1} - \varepsilon$. Call this agent $i^*(t)$, the agent with the highest type at t which is not $1 - \varepsilon$. It follows that

$$N^k_{k+1}(t) = N^k_k(t) \cup \{i^*(t)\}.$$

We will *claim* here that for every $k = 1, ..., n-1$ and every $t \in T^k$, $N^k_{k+1}(t) \subseteq S^g(t)$. This is a consequence of wFE and we will postpone the proof for a little while. Given this *claim*, the proof of Case 2 and therefore of Proposition 3 can easily be finished. We have for all $t \in T^{n-1}$

$$N = N^{n-1}_n(t) \subseteq S^g(t)$$

where the equality is by construction of the sets $N^k_{k+1}(t)$ and the set inclusion is by the claim. Since $S^g(t) \subseteq N$ as well by definition, $S^g(t) = N$ for all $t \in T^{n-1}$. By SP and IR, therefore

$$\begin{aligned} S^g(t^*) &= N \text{ and} \\ p^g_i(t^*) &\leq \frac{1}{n} - \varepsilon \text{ for all } i, \end{aligned}$$

leading to a budget deficit at t^*.

All that remains now is to prove the claim. We will use induction.

Let $k = 1$ and take any $t \in T^1$. Since $S^g(t)$ contains some agent, it contains the agent with type $1-\varepsilon$. In other words, $N_1^1(t) \subseteq S^g(t)$. However we can not have $N_1^1(t) = S^g(t)$ as this would either violate IR, or lead to a budget deficit. Hence $S^g(t)$ contains other agents and one of these must be $i^*(t)$ by wFE. Thus

$$N_1^1(t) \cup \{i^*(t)\} = N_2^1(t) \subseteq S^g(t).$$

Now suppose, as induction hypothesis, that for some $k \in \{2, ..., n\}$, $N_k^{k-1}(t) \subseteq S^g(t)$ for all $t \in T^{k-1}$. Take any $t \in T^k$. There exist, by construction of the sets $T^1, ..., T^n$, type vectors $t^1, ..., t^k \in T^{k-1}$ such that for every $l = 1, ..., k$

$$t_i = \begin{cases} t_i^l & \text{if } i \neq i^*(t^l)\text{, and} \\ 1-\varepsilon & \text{if } i = i^*(t^l). \end{cases}$$

It follows that

$$i^*(t^l) \in N_k^{k-1}(t^l) \subseteq S^g(t^l).$$

Furthermore by SP and IR

$$\begin{aligned} i^*(t^l) &\in S^g(t) \text{ and} \\ p_{i^*(t^l)} &\leq \frac{1}{k} - \varepsilon \text{ for every } l. \end{aligned}$$

Note that $\{i^*(t^l) : l = 1, ..., k\} = N_k^k(t)$, thus $N_k^k(t) \subseteq S^g(t)$. But $N_k^k(t)$ can not cover the cost of the club at t. Hence either there is budget deficit at t, or there is at least one more member in $S^g(t)$. By wFE this member is $i^*(t)$ and

$$N_k^k(t) \cup \{i^*(t)\} = N_{k+1}^k(t) \subseteq S^g(t)$$

proving the claim. This finishes the proof of Proposition 3. ■

Appendix 2: Perturbation of Equal Cost Sharing

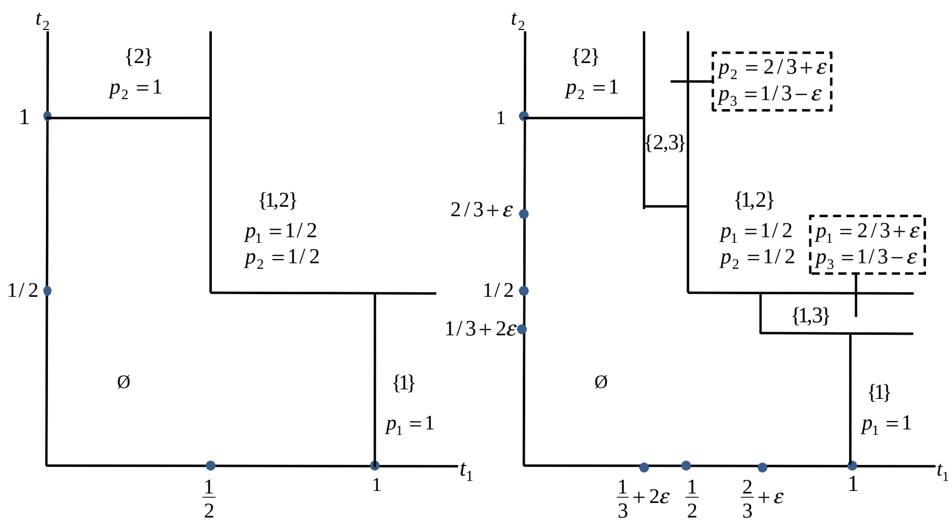
In this appendix, we will elaborate on the family of mechanisms f^{ε} which we introduced in Example 1. Suppose that $N = \{1, 2, 3\}$ and take $\varepsilon \in (0, \frac{1}{24})$. We would like to re-emphasize that $f^{\varepsilon} \to f$ pointwise as $\varepsilon \to 0$ and that f^{ε}, just like f, is an anonymous mechanism.

We will describe f^{ε} in a series of diagrams, each corresponding to a particular interval of t_3 values while t_1 and t_2 freely vary. To this end we will partition the joint type space $\Re^3_+$ into 6 parts, $T_1, ..., T_6$.

On

$$\begin{aligned} T_1 &= \{(t_1, t_2, t_3) \in \Re^3_+ : t_3 \in [0, \frac{1}{3} - \varepsilon)\}, \text{ and} \\ T_2 &= \{(t_1, t_2, t_3) \in \Re^3_+ : t_3 \in [\frac{1}{3} - \varepsilon, \frac{1}{3} + 2\varepsilon)\} \end{aligned}$$

f^{ε} is given by:



f^{ε} in T_1 $\qquad$ f^{ε} in T_2

Note that f^{ε} behaves identical to f in T_1. However this behavior ends at $t_3 = \frac{1}{3} - \varepsilon$. In particular $\sup_{t \in T_1} WL^{f^{\varepsilon}}(t) = \frac{5}{6} - \varepsilon$ and this supremum is given by type vectors $(1, \frac{1}{2}, \frac{1}{3} - \varepsilon)$ and $(\frac{1}{2}, 1, \frac{1}{3} - \varepsilon)$. Note that T_1 does not allow $WL^{f^{\varepsilon}}(t)$ to reach $MWL^f = \frac{5}{6}$ at any $t \in T_1$.

Next consider the space T_2 of type vectors. This is a rather thin slice as t_3 takes values in an interval of measure 3ε. On T_2, $\sup_{t \in T_2} WL^{f^{\varepsilon}}(t) = \frac{2}{3} + 4\varepsilon$

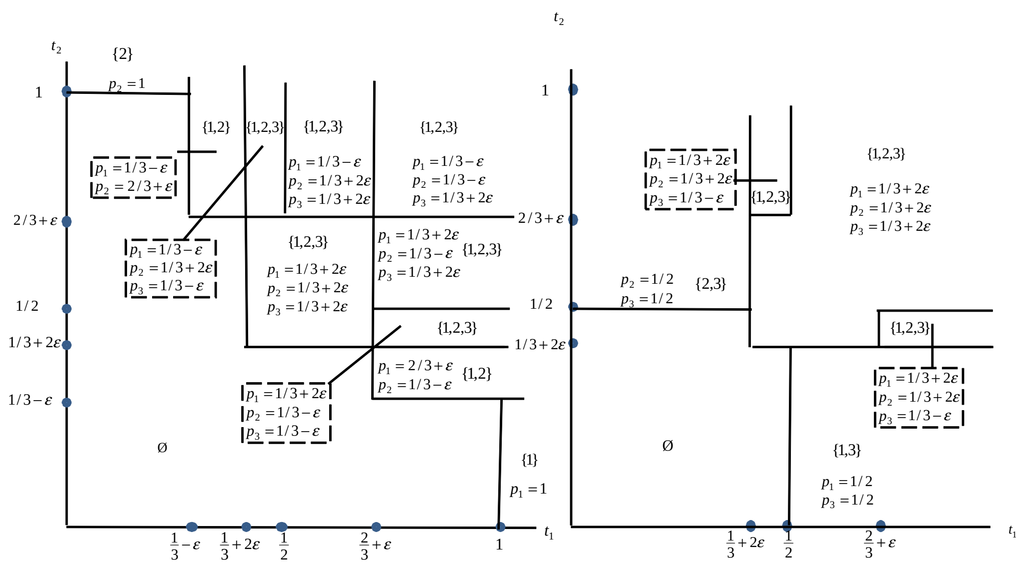
which falls short of $MWL^f = \frac{5}{6}$ as we took $\varepsilon < \frac{1}{24}$. The supremum on T_2 obtains close (from below) to the critical type vectors $(\frac{1}{3} + 2\varepsilon, 1, \frac{1}{3} + 2\varepsilon)$ and $(1, \frac{1}{3} + 2\varepsilon, \frac{1}{3} + 2\varepsilon)$.

Importantly, in T_2 clubs $\{2,3\}$ and $\{1,3\}$ are formed in order to cover the area where f obtains MWL. However at any t where $S(t) = \{2,3\}$, $t_1 \geq \frac{1}{3} + 2\varepsilon > t_3$ and at any t where $S(t) = \{1,3\}$, $t_2 \geq \frac{1}{3} + 2\varepsilon > t_3$. Hence f^ε excludes higher type agents from clubs at the benefit of lower type agents, and therefore violate wFE. Furthermore, f^ε also violates wDM: when all types increase the club formed could change from $\{2,3\}$ to $\{1,2\}$ or from $\{1,3\}$ to $\{1,2\}$. This is to be expected in light of Proposition 3. Any mechanism g such that $MWL^g < MWL^f$ needs to violate both wFE and wDM when $n > 2$.

We move on to higher values for t_3. On

$$
\begin{aligned}
T_3 &= \{(t_1, t_2, t_3) \in \Re^3_+ : t_3 \in [\frac{1}{3} + 2\varepsilon, \frac{1}{2})\} \text{ and} \\
T_4 &= \{(t_1, t_2, t_3) \in \Re^3_+ : t_3 \in [\frac{1}{2}, \frac{2}{3} + \varepsilon)\}
\end{aligned}
$$

f^ε is given by:



f^ε in T_3 $\qquad$ f^ε in T_4

Note that

$$\begin{aligned}
\sup_{t \in T_3} WL^{f^\varepsilon}(t) &= \frac{5}{6} - \varepsilon \text{ and} \\
\sup_{t \in T_4} WL^{f^\varepsilon}(t) &= \frac{1}{2} + 3\varepsilon
\end{aligned}$$

which are both strictly less than $MWL^f = \frac{5}{6}$. The supremum on T_3 occurs close to the critical type vectors $(\frac{1}{3}-\varepsilon, 1, \frac{1}{2})$ and $(1, \frac{1}{3}-\varepsilon, \frac{1}{2})$ and the supremum on T_4 occurs close to the critical type vectors $(\frac{1}{2}, \frac{1}{3}+2\varepsilon, \frac{2}{3}+\varepsilon)$ and $(\frac{1}{3}+2\varepsilon, \frac{1}{2}, \frac{2}{3}+\varepsilon)$.

Budget surpluses begin appearing in $T_3 \cup T_4$. The largest surplus is 6ε which appears in the set

$$[\frac{1}{3} + 2\varepsilon, \frac{2}{3} + \varepsilon) \times [\frac{1}{3} + 2\varepsilon, \frac{2}{3} + \varepsilon) \times [\frac{1}{3} + 2\varepsilon, \frac{2}{3} + \varepsilon).$$

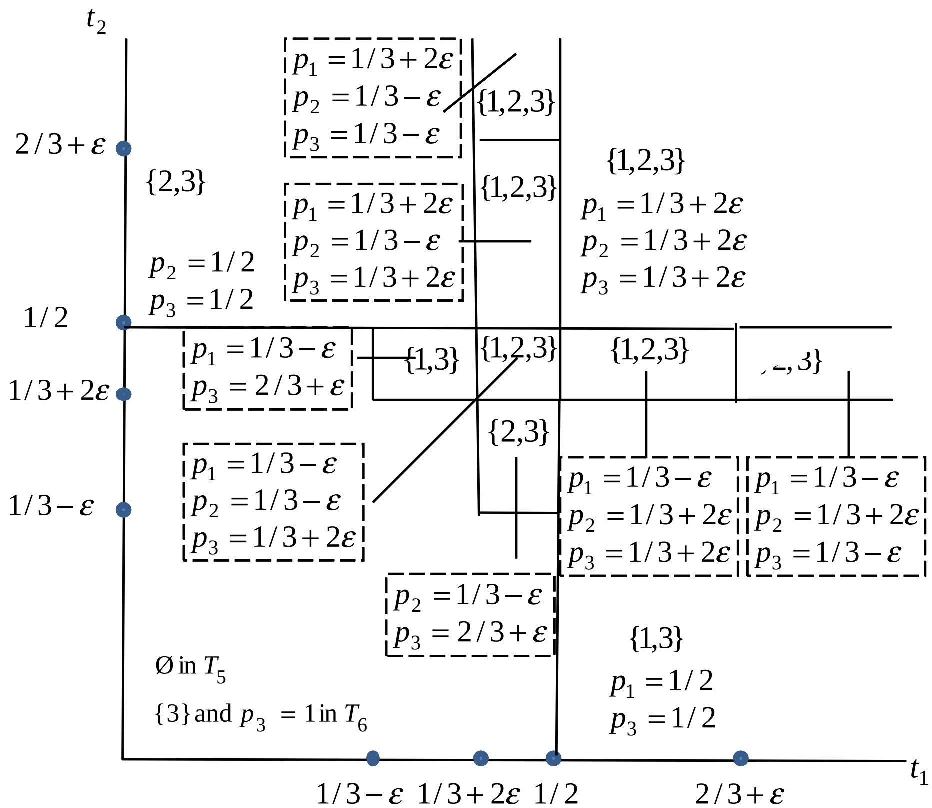
As we will see below in analyzing T_5 and T_6, these surpluses are necessary to avoid budget deficits at higher values for t_3, as we will see below.

Further, note that clubs $\{1, 2\}$ form in T_3 at type vectors where t_3 exceeds t_1 or t_2, indicating another failure of wFE. These clubs disappear as we move from T_3 to T_4. Take the club $\{1, 2\}$ which forms when $t_1 \geq \frac{2}{3} + \varepsilon$ and $t_2 \in [\frac{1}{3} - \varepsilon, \frac{1}{3} + 2\varepsilon)$ in T_3 for example. As t_3 increases beyond $\frac{1}{2}$ and we move on to T_4, this club disappears and in its stead, $\{1, 3\}$ forms. These are violations of wDM which are analogous to the violations that occur in T_2, with the names of the agents interchanged.

Finally on

$$\begin{aligned}
T_5 &= \{(t_1, t_2, t_3) \in \Re^3_+ : t_3 \in [\frac{2}{3} + \varepsilon, 1)\} \text{ and} \\
T_6 &= \{(t_1, t_2, t_3) \in \Re^3_+ : t_3 \in [1, \infty)\}
\end{aligned}$$

f^ε is given by:



f^{ε} in T_5 and T_6

The unique difference between T_5 and T_6 is that in T_5 the empty club is formed at low values of (t_1, t_2) whereas in T_6 where $t_3 \geq 1$, the singleton club $\{3\}$ forms. Note that

$$\sup_{t \in T_5} WL^{f^{\varepsilon}}(t) = \max\{\frac{5}{6} - \varepsilon, \frac{2}{3} + 4\varepsilon\} < \frac{5}{6} = MWL^{f}$$

as $\varepsilon < \frac{1}{24}$. Furthermore

$$\sup_{t \in T_6} WL^{f^{\varepsilon}}(t) = 6\varepsilon$$

and this is caused solely by budget surpluses as at no vector in T_6 does f^{ε} form the empty club.

The new clubs $\{1,3\}$ and $\{2,3\}$ which appear in T_5 serve the purpose of keeping the welfare loss below $\frac{5}{6}$. However they produce an important side effect. Take, for example, the club $\{1,3\}$ that forms when $\frac{1}{3}-\varepsilon \leq t_1 < \frac{1}{3}+2\varepsilon$ and $\frac{1}{3}+2\varepsilon \leq t_2 < \frac{1}{2}$. First note that this is a violation of wFE. Next, since agent 1 now has a lower cutoff, $\frac{1}{3}-\varepsilon$, his payment is lower in every club formed when his type increases and other two types remain constant. In particular when $t_1 > \frac{2}{3}+\varepsilon$, note that budget is exactly balanced. This gives the rationale for budget surplus in the previous plot in the same area when $t_3 \in [\frac{1}{2}, \frac{2}{3}+\varepsilon)$. The same holds when we start from a type vector when $\{2,3\}$ forms and increase t_2, keepong t_1 and t_3 constant.

Note that equating $\frac{5}{6}-\varepsilon = \frac{2}{3}+4\varepsilon$ we find $\varepsilon = \frac{1}{30}$, the value for ε which induces the lowest maximal welfare loss, $\frac{4}{5}$, in the class $\{f^{\varepsilon} : 0 < \varepsilon < \frac{1}{24}\}$.